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On the spectral moment of quasi-trees ☆

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ABSTRACT

A connected graph $G = (V, E)$ is called a quasi-tree, if there exists $u_0 \in V(G)$ such that $G - u_0$ is a tree. Denote $\mathcal{Q}(n, d_0) = \{G : G$ is a quasi-tree graph of order n with $G - u_0$ being a tree and $d_G(u_0) = d_0\}$. Let $A(G)$ be the adjacency matrix of a graph G, and let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ be the eigenvalues in non-increasing order of $A(G)$. The number $\sum_{i=1}^{n} \lambda_i^k(G)$ $(k = 0, 1, \ldots, n-1)$ is called the kth spectral moment of G, denoted by $S_k(G)$. Let $S(G) = (S_0(G), S_1(G), \ldots, S_{n-1}(G))$ be the sequence of spectral moments of G. For two graphs G_1, G_2, we have $G_1 \prec_S G_2$ if for some k $(k = 1, 2, \ldots, n-1)$, we have $S_i(G_1) = S_i(G_2)$ $(i = 0, 1, \ldots, k-1)$ and $S_k(G_1) < S_k(G_2)$. In this paper, we determine the last and the second last quasi-tree, in an S-order, in the set $\mathcal{Q}(n, d_0)$, respectively.

1. Introduction

All graphs considered here are finite and simple. Undefined terminology and notation may refer to [1]. Let $G = (V, E)$ be a simple undirected graph with n vertices. For $v \in V(G)$, we use $N_G(v)$ to denote the neighbor of v in graph G and set $d_G(v) = |N_G(v)|$. A leaf of a graph is a vertex of degree one. We will use $G - x$ or $G - xy$ to denote the graph that arises from G by deleting the vertex $x \in V(G)$ or the edge $xy \in E(G)$. Similarly, $G + xy$ is a graph that arise from G by adding an edge $xy \notin E(G)$, where $x, y \in V(G)$. If there is only one vertex $u \in V(T)$ with $d_T(u) = t \geqslant 3$ and $d_T(v) \leqslant 2$ for $v \in V(T) \setminus \{u\}$, then we call T a t-tree.

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Let $A(G)$ be the adjacency matrix of a graph G, and let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ be the eigenvalues in non-increasing order of a graph G. The number $\sum_{i=1}^{n} \lambda_i^k(G)$ $(k = 0, 1, \ldots, n-1)$ is called the kth spectral moment of G, denoted by $S_k(G)$. Note that $S_0 = n$, $S_1 = l$, $S_2 = 2m$, $S_3 = 6t$, where n, l, m, t denote the number of vertices, the number of loops, the number of edges and the number of triangles respectively (see [2]). Let $S(G) = (S_0(G), S_1(G), \ldots, S_{n-1}(G))$ be the sequence of spectral moments of G. For two graphs G_1, G_2, we shall write $G_1 =_S G_2$ if $S_i(G_1) = S_i(G_2)$ for $i = 0, 1, \ldots, n-1$. Similarly, we have $G_1 \prec_S G_2$ (G_1 comes before G_2 in an S-order) if for some k $(1 \le k \le n-1)$, we have $S_i(G_1) = S_i(G_2) (i = 0, 1, \ldots, k-1)$ and $S_k(G_1) < S_k(G_2)$. We shall also write $G_1 \preceq_S G_2$ if $G_1 \prec_S G_2$ or $G_1 =_S G_2$. S-order had been used in producing graph catalogs (see [4]), and for a more general setting of spectral moments, see [3].

Up to now, few results on the S-order of graphs are obtained. Cvetković and Rowlinson [5] studied the S-order of trees and unicyclic graphs and characterized the first and the last graphs, in an S-order, of all trees and all unicyclic graph with given girth, respectively. Wu and Fan [9] determined the first and the last graphs, in an S-order, of all unicyclic graphs and bicyclic graphs, respectively. Pan et al. [6] gave the first $\sum_{k=1}^{\lfloor \frac{n-1}{3} \rfloor} \left(\lfloor \frac{n-k-1}{2} \rfloor - k + 1 \right)$ graphs apart from a path, in an S-order, of all trees with n vertices. In [7] and [8], Wu and Liu determined the last $\lfloor \frac{d}{2} \rfloor + 1$ and the last $\lfloor \frac{g}{2} \rfloor + 2$, in an S-order, among all trees of order n and diameter d $(4 \le d \le n-3)$ and all unicyclic graphs of order n and girth g $(3 \le g \le n-3)$, respectively.

A connected graph G is called a *quasi-tree* if there exists $u_0 \in V(G)$ such that $G - u_0$ is a tree. Let $\mathcal{Q}(n, d_0)$={G : G is a quasi-tree graph of order n with $G - u_0$ being a tree and $d_G(u_0) = d_0$}. Then $d_0 \geqslant 1$.

In this paper, we will present the last and the second last, in an S-order, of quasi-trees in the set $\mathcal{Q}(n, d_0)$ $(1 \le d_0 \le n-1)$.

We first give some lemmas that will be used in the proof of our results.

Lemma 1.1 [2]. *The kth spectral moment of G is equal to the number of closed walks of length k.*

Let F be a graph. An F-subgraph of G is a subgraph of G which is isomorphic to the graph F. Let $\phi_G(F)$ (or $\phi(F)$) be the number of all F-subgraphs of G.

Let P_n, C_n, S_n be a path, a cycle and a star $K_{1,n-1}$ of order n, respectively. Let S_n^* be a graph obtained from a star S_{n-1} by attaching a leaf to one leaf of S_{n-1}. Let U_n be a graph obtained from C_{n-1} by attaching a leaf to one vertex of C_{n-1}.

Lemma 1.2. *For every graph G, we have*

(i) $S_4(G) = 2\phi(P_2) + 4\phi(P_3) + 8\phi(C_4)$ *(see [4]);*
(ii) $S_5(G) = 30\phi(C_3) + 10\phi(U_4) + 10\phi(C_5)$ *(see [7]).*

Let Q_{n,d_0} be a graph obtained from a star S_{n-1} and an isolated vertex u_0 by adding an edge joining u_0 to the center of S_{n-1}, and $d_0 - 1$ edges joining u_0 to the leaves of S_{n-1}, respectively. Then $Q_{n,1} \cong S_n$.

Let $Q_{n,1}^* \cong S_n^*$, and let Q_{n,d_0}^* $(2 \le d_0 \le n-2)$ be a graph obtained from a graph Q_{n-1,d_0} by attaching a leaf to one vertex of degree 2 in Q_{n-1,d_0}, and let $Q_{n,n-1}^*$ be a graph obtained from a graph S_{n-1}^* and an isolated vertex u_0 by adding $n-1$ edges joining u_0 to each vertex of S_{n-1}^*.

Let Q'_{n,d_0} $(1 \le d_0 \le n-2)$ be a graph obtained from a star S_{n-1} and an isolated vertex u_0 by adding d_0 edges joining u_0 to the leaves of S_{n-1}. Then $Q'_{n,1} \cong S_n^*$, and $Q_{n,d_0}, Q_{n,d_0}^*, Q'_{n,d_0} \in \mathcal{Q}(n, d_0)$. For example, $Q_{6,d_0}, Q_{6,d_0}^*, d_0 = 2, 3, 4, 5$ and $Q'_{6,d_0}, d_0 = 2, 3, 4$ are shown in Fig. 1, respectively.

2. The last quasi-tree in $\mathcal{Q}(n, d_0)$

In this section, we always assume that $G \in \mathcal{Q}(n, d_0)$ $(n \ge 4)$ with $V(G) = \{u_0, u_1, \ldots, u_{n-1}\}$. Denote $G' = G - u_0$. Choose a vertex $u_1 \in V(G')$ such that $d_{G'}(u_1) = \Delta(G') \ge 2$.

Lemma 2.1. *Suppose that $u_0u_1 \notin E(G)$, then there is a graph $G^* \in \mathcal{Q}(n, d_0)$ such that $G \prec_S G^*$.*

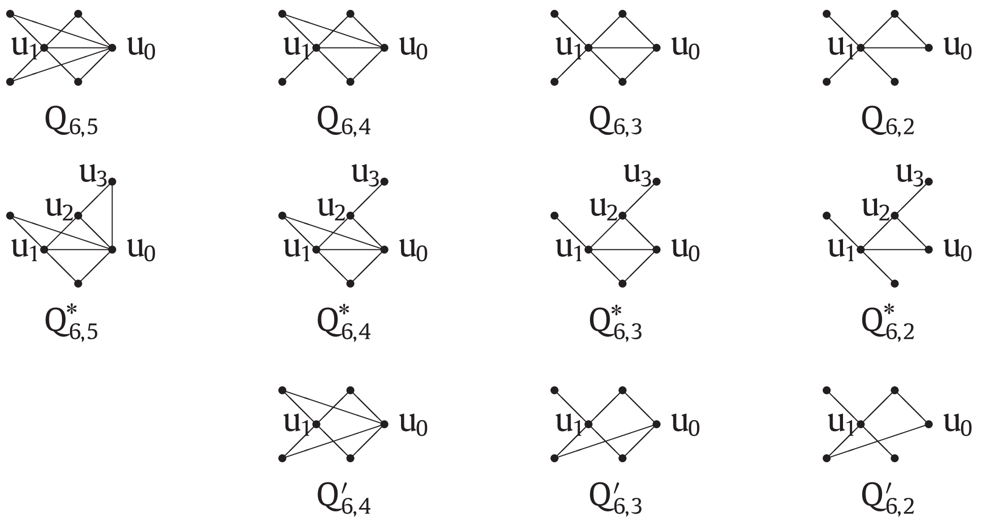

Fig. 1.

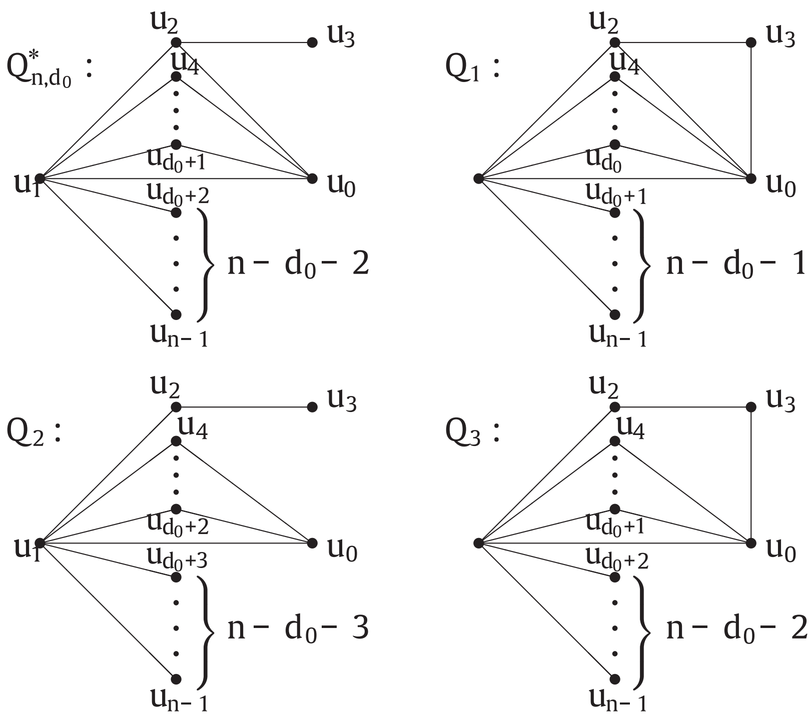

Fig. 2.

Proof. Since $d_0 \geqslant 1$, there is a vertex $u_i \in V(G') \setminus \{u_1\}$ $(i \geq 2)$ such that $u_0u_i \in E(G)$. Let $d_{G'}(u_1) = k$ and $d_{G'}(u_i) = l$, then $k \geqslant 2$ and $l \geqslant 1$. Since G' is a tree, there is a unique (u_1, u_i)-path Q in G'. Denote $N_{G'}(u_1) = \{v_1, v_2, \ldots, v_k\}$. Without loss of generality, we assume that $v_1 \in V(Q)$. Set

$$G^* = G - \{u_1v_2, u_1v_3, \ldots, u_1v_k\} + \{u_iv_2, u_iv_3, \ldots, u_iv_k\}.$$

Then $G^* \in \mathscr{Q}(n, d_0)$. Now we will show that $G \prec_S G^*$.

If $d_0 = 1$, then $S_i(G) = S_i(G^*)$ $(i = 1, 2, 3)$. Since $k \geqslant 2$, $l \geqslant 1$, then

$$\phi_G(P_3) - \phi_{G^*}(P_3) = \binom{k}{2} + \binom{l+1}{2} - \binom{k+l}{2} = -l(k-1) < 0.$$

Note that $\phi_G(C_4) = \phi_{G^*}(C_4) = 0$. Hence, by Lemma 1.2, we have

$$S_4(G) - S_4(G^*) = 4(\phi_G(P_3) - \phi_{G^*}(P_3)) + 8(\phi_G(C_4) - \phi_{G^*}(C_4)) < 0.$$

Thus $G \prec_S G^*$.

Therefore, we assume that $d_0 \geqslant 2$. Set $p = |N_G(u_0) \cap \{v_2, \ldots, v_k\}|$. Note that $S_i(G) = S_i(G^*)(i = 1, 2)$. We consider two cases.

Case 1. $N_G(u_0) \cap \{v_2, \ldots, v_k\} \neq \emptyset$.

In this case, $p \geq 1$. Note that $u_0u_iv_ju_0$ is a triangle in G^* but not in G for any $v_j \in N_G(u_0) \cap \{v_2, \ldots, v_k\}$. Moreover, if G contains a triangle, then there is the same triangle other than $u_0u_iv_ju_0$ in G^*, and vice versa. Hence

$$S_3(G) - S_3(G^*) = -6p < 0.$$

Thus $G \prec_S G^*$.

Case 2. $N_G(u_0) \cap \{v_2, \ldots, v_k\} = \emptyset$.

In this case, we set $q = \sum_{i=2}^{k} |N_G(u_0) \cap N_G(v_i)|$ and note that

$$S_3(G) = S_3(G^*) \text{ and } \phi_G(P_3) - \phi_{G^*}(P_3) = -l(k-1) < 0.$$

Suppose $N_G(u_0) \cap N_G(v_j) \neq \emptyset$, and let $w \in N_G(u_0) \cap N_G(v_j)$ for some $2 \leq j \leq k$. Then $u_0u_iv_jwu_0$ is a 4-cycle C_4 in G^* but not in G. Note that if $u_1v_aw_{ab}v_bu_1$ is a 4-cycle in G, then $u_iv_aw_{ab}v_bu_i$ is also a 4-cycle in G^*, where $2 \leq a, b \leq k$ and $w_{ab} \in N(v_a) \cap N(v_b)$. Moreover, if G contains a 4-cycle $C_4 \neq u_1v_aw_{ab}v_bu_1$, then there is the same 4-cycle other than $u_iv_aw_{ab}v_bu_i$ in G^*, and vice versa. So $\phi_G(C_4) - \phi_{G^*}(C_4) = -q \leqslant 0$.
Hence, by Lemma 1.2, we have

$$S_4(G) - S_4(G^*) = 4(\phi_G(P_3) - \phi_{G^*}(P_3)) + 8(\phi_G(C_4) - \phi_{G^*}(C_4)) < 0. \quad \text{Thus } G \prec_S G^*. \quad \square$$

In order to get the last graph in an S-order of all the quasi-trees, from Lemma 2.1, we only need to discuss those graphs in $G \in \mathscr{Q}(n, d_0)$ with $u_0u_1 \in E(G)$.

Denote $\mathscr{Q}'(n, d_0) = \{G : G \in \mathscr{Q}(n, d_0),\ u_0u_1 \in E(G)\}$. Then $Q_{n,d_0} \in \mathscr{Q}'(n, d_0)$.

Theorem 2.2. *Let $G \in \mathscr{Q}'(n, d_0) \setminus \{Q_{n,d_0}\}$. Then $G \prec_S Q_{n,d_0}$.*

Proof. First we note that G' is not a star. Hence there exists a vertex $u_2 \in N_{G'}(u_1)$ such that $d_{G'}(u_2) \geq 2$. Let $N_{G'}(u_2) \setminus \{u_1\} = \{v_1, v_2, \ldots, v_a\}$, then $d_{G'}(u_2) = a + 1$ and $a \geq 1$. Suppose $d_{G'}(u_1) = k \geq 2$. Then $k \geq a + 1$. Let

$$G^* = G - \{u_2v_1, u_2v_2, \ldots, u_2v_a\} + \{u_1v_1, u_1v_2, \ldots, u_1v_a\}.$$

Then $G^* \in \mathscr{Q}'(n, d_0)$. Next we will show that $G \prec_S G^*$. Note that $S_i(G) = S_i(G^*)$ $(i = 1, 2)$. We consider the following two cases.

Case 1. $u_0u_2 \notin E(G)$.

If there exist v_i such that $u_0v_i \in E(G)$ for some $1 \leqslant i \leqslant a$, then $u_0v_iu_1u_0$ is a triangle in G^* but not in G. Moreover, if G contains a triangle, then there is the same triangle other than $u_0v_iu_1u_0$ in G^*, and vice versa. So we have

$$S_3(G) - S_3(G^*) < 0.$$

Thus $G \prec_S G^*$. Therefore we assume that $u_0v_i \notin E(G)$ for all $i,\ 1 \leqslant i \leqslant a$. Then $S_3(G) = S_3(G^*)$. In the following, we consider the number of C_4 and P_3 in G and G^*, respectively. It is easy to see that if $w \in N_G(u_0) \cap N_G(v_i)$ for some $i,\ 1 \leqslant i \leqslant a$, then $u_0u_1v_iwu_0$ is a 4-cycle in G^* but not in G. Moreover, if G contains a 4-cycle C_4, then there is the same 4-cycle other than $u_0u_1v_iwu_0$ in G, and vice versa. So $\phi_G(C_4) - \phi_{G^*}(C_4) \leqslant 0$. Note that

$$\phi_G(P_3) - \phi_{G^*}(P_3) = \binom{k+1}{2} + \binom{a+1}{2} - \binom{k+a+1}{2} = -ak < 0.$$

Hence, by Lemma 1.2, we have

$$S_4(G) - S_4(G^*) = 4(\phi_G(P_3) - \phi_{G^*}(P_3)) + 8(\phi_G(C_4) - \phi_{G^*}(C_4)) < 0.$$

Thus $G \prec_S G^*$.

Case 2. $u_0u_2 \in E(G)$.

In this case, we consider two subcases.

Subcase 2.1. $u_0v_i \notin E(G)$ for all $i,\ 1 \leqslant i \leqslant a$.

In this subcase, we first note that $S_3(G) = S_3(G^*)$, and if $w \in N_G(u_0) \cap N_G(v_i)$ for some $i,\ 1 \leqslant i \leqslant a$, then $u_0u_2v_iwu_0$ is a 4-cycle in G not in G^*, and $u_0u_1v_iwu_0$ is also a 4-cycle in G^* not in G. Moreover, if G contains a 4-cycle $C_4 \neq u_0u_2v_iwu_0$, then there is the same 4-cycle other than $u_0u_1v_iwu_0$ in G^*, and vice versa. So we have $\phi_G(C_4) = \phi_{G^*}(C_4)$. Note that

$$\phi_G(P_3) - \phi_{G^*}(P_3) = \binom{k+1}{2} + \binom{a+2}{2} - \binom{k+a+1}{2} - \binom{2}{2} = a(1-k) < 0.$$

Hence, by Lemma 1.2, we have

$$S_4(G) - S_4(G^*) = 4(\phi_G(P_3) - \phi_{G^*}(P_3)) + 8(\phi_G(C_4) - \phi_{G^*}(C_4)) < 0.$$

Thus $G \prec_S G^*$.

Subcase 2.2. $u_0v_i \in E(G)$ for some $i,\ 1 \leqslant i \leqslant a$.

In this subcase, $u_0u_2v_iu_0$ is a triangle in G not in G^* and $u_0u_1v_iu_0$ is a triangle in G^* not in G^*. Moreover, if G contains a triangle $C_3 \neq u_0u_2v_iu_0$, then there is the same triangle other than $u_0u_1v_iu_0$ in G^*, and vice versa. Thus $S_3(G) = S_3(G^*)$.

Note that $u_0v_iu_2u_1u_0$ is a 4-cycle in G not in G^* and $u_0v_iu_1u_2u_0$ is a 4-cycle in G^* not in G. On the other hand, if there is a vertex $w \in N_G(u_0) \cap N_G(v_t)$ for some $t,\ 1 \leqslant t \leqslant a$, then $u_0u_2v_twu_0$ is a 4-cycle in G not in G^* and $u_0u_1v_twu_0$ is also a 4-cycle in G^* not in G. Moreover, if G contains a 4-cycle $C_4 \neq u_0u_2v_twu_0$ and $C_4 \neq u_0v_tu_2u_1u_0$, then there is the same 4-cycle other than $u_0u_1v_twu_0$ and $u_0v_iu_1u_2u_0$ in G^*, and vice versa. So we have $\phi_G(C_4) = \phi_{G^*}(C_4)$. Note that

$$\phi_G(P_3) - \phi_{G^*}(P_3) = \binom{k+1}{2} + \binom{a+2}{2} - \binom{k+a+1}{2} - \binom{2}{2} = a(1-k) < 0.$$

Hence $G \prec_S G^*$.

Finally, if $G^* \cong Q_{n,d_0}$, then $G \prec_S G^* =_S Q_{n,d_0}$. Otherwise, repeating the above discussion, we have $G \prec_S G^* \prec_S \ldots =_S Q_{n,d_0}$. $\square$

By Lemma 2.1 and Theorem 2.2, we have

Theorem 2.3. *For any $G \in \mathcal{Q}(n, d_0) \setminus \{Q_{n,d_0}\}$, we have*

$$G \prec_S Q_{n,d_0}.$$

3. The second last quasi-tree in $\mathscr{Q}(n, d_0)$

In this section, we will present the last quasi-tree in $G \in \mathscr{Q}(n, d_0) \setminus \{Q_{n,d_0}\}$. Let Q_1, Q_2, Q_3 be three graphs in $\mathscr{Q}(n, d_0)$ as follows (see Fig. 2).

Proposition 3.1. *Let Q'_{n,d_0}, Q^*_{n,d_0} and $Q_j,\ j = 1, 2, 3$ be five graphs defined above. Then*

(i) $Q'_{n,d_0} \prec_S Q^*_{n,d_0}, Q_2 \prec_S Q^*_{n,d_0},\ \ Q_3 \prec_S Q^*_{n,d_0}$ *for* $d_0 \geq 2$;
(ii) $Q_1 \prec_S Q^*_{n,d_0}$ *for* $d_0 \geq 3$ *and* $n \geq 6$.

Proof. Note that $S_i(Q'_{n,d_0}) = S_i(Q^*_{n,d_0}) = S_i(Q_j)$ for $i = 1, 2$ and $j = 1, 2, 3$.

(i) Since $S_3(Q'_{n,d_0}) = 0 < S_3(Q^*_{n,d_0})$ for $d_0 \geq 2$, we have $Q'_{n,d_0} \prec_S Q^*_{n,d_0}$.

Note that $S_3(Q_2) = S_3(Q^*_{n,d_0})$, $\phi_{Q^*_{n,d_0}}(C_4) - \phi_{Q_2}(C_4) = \binom{d_0}{2} - \binom{d_0}{2} = 0$ and

$$\phi_{Q^*_{n,d_0}}(P_3) - \phi_{Q_2}(P_3) = \binom{3}{2} - \binom{2}{2} = 1 > 0.$$

By Lemma 1.2, we have

$$S_4(Q^*_{n,d_0}) - S_4(Q_2) = 4\left(\phi_{Q^*_{n,d_0}}(P_3) - \phi_{Q_2}(P_3)\right) + 8\left(\phi_{Q^*_{n,d_0}}(C_4) - \phi_{Q_2}(C_4)\right) > 0.$$

Thus $Q_2 \prec_S Q^*_{n,d_0}$.

Clearly, $S_3(Q^*_{n,d_0}) - S_3(Q_3) = 1 > 0$. Thus $Q_3 \prec_S Q^*_{n,d_0}$.

(ii) Note that $S_3(Q_1) = S_3(Q^*_{n,d_0})$ and $\phi_{Q^*_{n,d_0}}(P_3) - \phi_{Q_1}(P_3) = 0$. If $d_0 \geq 4$, then

$$\phi_{Q^*_{n,d_0}}(C_4) - \phi_{Q_1}(C_4) = \binom{d_0 - 1}{2} - \left(\binom{d_0 - 2}{2} + 1\right) = d_0 - 3 > 0. \qquad (*)$$

By Lemma 1.2, we have

$$S_4(Q^*_{n,d_0}) - S_4(Q_1) = 4(\phi_{Q^*_{n,d_0}}(P_3) - \phi_{Q_1}(P_3)) + 8(\phi_{Q^*_{n,d_0}}(C_4) - \phi_{Q_1}(C_4)) > 0.$$

Thus $Q_1 \prec_S Q^*_{n,d_0}$.

If $d_0 = 3$ and $n \geq 6$, then $S_4(Q^*_{n,d_0}) = S_4(Q_1)$, $\phi_{Q_1}(C_5) = \phi_{Q^*_{n,d_0}}(C_5) = 0$ and

$$\phi_{Q_1}(U_4) - \phi_{Q^*_{n,d_0}}(U_4) = (n - d_0 - 1) - 2(n - d_0 - 2) - 1 = -(n - d_0 - 2) < 0.$$

Thus, by Lemma 1.2, $S_5(Q_1) - S_5(Q^*_{n,d_0}) < 0$. Hence $Q_1 \prec_S Q^*_{n,d_0}$. $\square$

In the following, we always assume that $G \in \mathscr{Q}(n, d_0) \setminus \{Q_{n,d_0}\}$, $n \geq 5$, $1 \leq d_0 \leq n - 1$ with $V(G) = \{u_0, u_1, \ldots, u_{n-1}\}$. Denote $G' = G - u_0$. Choose a vertex $u_1 \in V(G')$ such that $d_{G'}(u_1) = \Delta(G') \geq 2$.

Lemma 3.2. *Suppose that $u_0u_1 \notin E(G)$. Then there is a graph $G^{**} \in \mathscr{Q}(n, d_0) \setminus \{Q_{n,d_0}\}$ such that $G \preceq_S G^{**}$, and further, $G \prec_S G^{**}$ when $d_0 \geq 2$.*

Proof. Since G is connected and $d_0 \geqslant 1$, there is a vertex $u_i \in V(G') \setminus \{u_1\}$ $(i \geq 2)$ such that $u_0u_i \in E(G)$. Since G' is a tree, there is a unique (u_1, u_i)-path Q in G'. Choose u_i such that $d_{G'}(u_i)$ is as large as possible. If $d_{G'}(u_i) = 1$ and $G' \cong S_{n-1}$, then $G \cong Q'_{n,d_0}$, and thus $G =_S Q'_{n,d_0} =_S Q^*_{n,d_0}$ for $d_0 = 1$, $G =_S Q'_{n,d_0} \prec_S Q^*_{n,d_0}$ for $d_0 \geq 2$. Thus we may assume that $d_{G'}(u_i) \geq 2$ or $d_{G'}(u_i) = 1$

and $G' \ncong S_{n-1}$. Denote $N_{G'}(u_1) = \{v_1, v_2, \ldots, v_k\}$, $k \geq 2$. Without loss of generality, we assume that $v_1 \in V(Q)$. Set

$$G^{**} = \begin{cases} G - \{u_1v_3, \ldots, u_1v_k\} + \{u_iv_3, \ldots, u_iv_k\}, & \text{if } d_{G'}(u_i) \geq 2, \\ G - \{u_1v_2, \ldots, u_1v_k\} + \{u_iv_2, \ldots, u_iv_k\}, & \text{if } d_{G'}(u_i) = 1 \text{ and } G' \ncong S_{n-1}. \end{cases}$$

Then $G^{**} \in \mathcal{Q}(n, d_0) \backslash \{Q_{n,d_0}\}$. By an argument similar to the proof of Lemma 2.1, we have $G \prec_S G^{**}$. □

Theorem 3.3. *Let $G \in \mathcal{Q}(n, d_0) \setminus \{Q_{n,d_0}, Q^*_{n,d_0}\}$. Then $G \prec_S Q^*_{n,d_0}$.*

Proof. By Lemma 3.2, we can assume that $u_0u_1 \in E(G)$.

Since $G \ncong Q_{n,d_0}$, G' is not a star. If $G' \cong S^*_{n-1}$, then $G \cong Q_1$ (when $n \geq 6$ and $d_0 \geq 3$), or $G \cong Q_2$, or $G \cong Q_3$ as $G \ncong Q^*_{n,d_0}$. Thus, by Proposition 3.1, $G =_S Q_i \prec_S Q^*_{n,d_0}$, $i = 1, 2, 3$. Therefore we assume that $G' \ncong S^*_{n-1}$.

If there is a vertex $u_i \in V(G)$ $(i \geq 2)$ such that $d_{G'}(u_i) \geq 3$. Denote $N_{G'}(u_i) = \{v_1, v_2, \ldots, v_a\}$, where $a \geq 3$, v_1 is the unique vertex belonging to the unique (u_1, u_i)-path in G'. Let

$$G^{**} = G - \{u_iv_3, \ldots, u_iv_a\} + \{u_1v_3, \ldots, u_1v_a\}.$$

Then $G^{**} \in \mathcal{Q}(n, d_0) \setminus \{Q_{n,d_0}, Q^*_{n,d_0}\}$. By an argument similar to the proof of Theorem 2.2, we have $G \prec_S G^{**}$. Therefore we may assume that G' is a k-tree.

Let $V^* = \{u : u \in V(G'), d_{G'}(u) = 2\}$ and $t = |V^*|$. Then $t \geq 2$ as $G' \ncong S^*_{n-1}$. Let $u_2 \in V^*$ with $N_{G'}(u_2) = \{u_1, u_3\}$. Let $G^{**} = G - u_2u_3 + u_1u_3$. Then $G^{**} \in \mathcal{Q}(n, d_0) \setminus \{Q_{n,d_0}\}$. By an argument similar to the proof of Theorem 2.2, we have $G \prec_S G^{**}$. If $t = 2$, then $G^{**} \cong Q^*_{n,d_0}$, and hence the result holds. If $t \geq 3$, then repeating the above, we can get a sequence of graphs $G^{3*}, \ldots, G^{(t-1)*} \in \mathcal{Q}(n, d_0) \setminus \{Q_{n,d_0}, Q^*_{n,d_0}\}$ and $G^{t*} \cong Q^*_{n,d_0}$ such that

$$G \prec_S G^{**} \prec_S G^{3*} \prec_S \cdots \prec_S G^{(t-1)*} \prec_S G^{t*} =_S Q^*_{n,d_0}.$$

Therefore the proof of the theorem is complete. □

Note that $\mathcal{Q}(n, 1)$ is the set of all trees, and hence by Theorems 2.3 and 3.3, we have the following result.

Corollary 3.4. *For any tree T of order n, we have*

(i) *$T \preceq_S S_n$ and $T =_S S_n$ if and only if $T \cong S_n$ (see [5]);*
(ii) *if $T \ncong S_n$, then $T \preceq_S S^*_n$ and $T =_S S^*_n$ if and only if $T \cong S^*_n$.*

Note that if we add an edge e to a connected graph G, then $G \prec_S G + e$ as $|E(G + e)| > |E(G)|$. So we have the following result.

Proposition 3.5. $Q_{n,d_0} \prec_S Q_{n,d_0+1}$ *and* $Q^*_{n,d_0} \prec_S Q^*_{n,d_0+1}$ *for* $1 \leq d_0 \leq n - 2$.

By Proposition 3.5, Theorems 2.3 and 3.3, we have the following results.

Theorem 3.6. *Let G be a quasi-tree of order n. Then $G \preceq_S Q_{n,n-1}$ and $G =_S Q_{n,n-1}$ if and only if $G \cong Q_{n,n-1}$. Moreover, if $G \ncong Q_{n,n-1}$, then $G \preceq_S Q^*_{n,n-1}$ and $G =_S Q^*_{n,n-1}$ if and only if $G \cong Q^*_{n,n-1}$.*

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